

# RANDOM GRAPH: STRONGER LOGIC BUT WITH THE ZERO ONE LAW

SAHARON SHELAH

ABSTRACT. We like to find a logic stronger than first order such that: on the one hand it satisfies the 0-1 law, e.g. for the random graph  $\mathcal{G}_{n,1/2}$  and on the other hand there is a formula  $\varphi(x)$  such that for no first order  $\psi(x)$  do we have: for every random enough  $G_{n,1/2}$  are the formulas  $\varphi(x), \psi(x)$  are equivalent in it.

---

*Date:* October 27, 2015.

*2010 Mathematics Subject Classification.* Primary: 03C13; Secondary:

*Key words and phrases.* finite model theory, random graphs, 0-1 laws, logics for finite models.

The author thanks Alice Leonhardt for the beautiful typing. Publication 1077.

## Anotated Content

## §0 Introduction

## §1 Identifying the too simple graphs

[We choose a  $\mathbf{h} : \mathbb{N} \rightarrow (0, 1)_{\mathbb{R}}$  going to zero slowly enough. Our intention is to add to first-order logic a quantifier describing random properties of a graph but excluding some “low”, “explicitly not random” graph. Those are graphs such that for any quantifier free first order formula  $\varphi(\bar{x}_0, \bar{x}_1, \bar{z})$  for some  $k$ , for random enough  $G = \mathcal{G}_{n, 1/2}$  (or  $\mathcal{G}_{n, p}$  for a given  $p \in (0, 1)_{\mathbb{R}}$ ), if  $\bar{c} \in {}^{\ell g(\bar{z})}G$  and  $\varphi(\bar{x}_0, \bar{x}_1, \bar{c})$  define in  $G$  a graph with  $> k$  nodes then it is so called low. This will be used in §2 to find a logic as desired.]

## §2 The Quantifier

[We choose randomly enough set  $\mathbf{K}$  of (isomorphism types of) finite non- $\mathbf{k}$ -low graphs and show that adding a quantifier for it preserves the zero-one law. In the “randomly”, the probability of a  $H$ , a non-low graph to be in the class is  $\mathbf{h}(|H|)$ . Why  $\mathbf{h}$  is not constant? Because we like  $\Pr(\mathcal{G}_{n, p} \in \mathbf{K})$  to converge to 0 (or to 1).]

## § 0. INTRODUCTION

Our aim is to find a logic  $\mathcal{L}$  stronger than first order such that: for  $p \in (0, 1)_{\mathbb{R}}$ , the  $p$ -random graph  $\mathcal{G} = \mathcal{G}_{n,p}$  (i.e. with edge probability  $p$ ) satisfies the 0-1-law but some formula  $\varphi(x) \in \mathbb{L}(\text{graphs})$  defines in random enough graph  $\mathcal{G}_{n,p}$  a set of nodes not definable by any first order logic formula (of course, small enough compared to  $n$ , even with parameters).

The logic is gotten from first order  $\mathbb{L}$  by adding a (Lindström) quantifier  $\mathbb{Q}_{\bar{\mathbf{t}}} = \mathbb{Q}_{K_{\bar{\mathbf{t}}}}$  gotten from a “random enough”  $\bar{\mathbf{t}} \in {}^{\mathbb{N}}\{0, 1\}$ ; on quantifiers see [Be85]. We may wonder, can we replace  $\mathbb{Q}$  by a “reasonably defined quantifier”? We may from the proof see what we need from  $\mathbf{K}$ , the class defining the quantifier  $\mathbf{Q}_{\mathbf{K}}$ , i.e. a class of (finite) graphs closed under isomorphisms. Excluding some graphs which we call low, the membership in  $\mathbf{K}$  will be random enough in the sense that if we consider only random enough  $\mathcal{G}_{n,p}$ , the  $\mathbb{L}(\mathbf{Q}_{\mathbf{K}})$ -formulas with parameters will define graphs which are not low and are pairwise non-isomorphic except in trivial cases. So we just need a definition satisfying this; we intend to do it in a work in preparation.

How does the randomness of  $\bar{\mathbf{t}}$  help us to get the zero-one law? The idea is that for the quantifier  $\mathbf{Q}_{\bar{\mathbf{t}}}$  (see §2) used here, if we expand  $\mathcal{G}_{n,p}$  by finitely many relations definable by formulas from  $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$ , we get a random structure with more relations essentially with constant probabilities, i.e. is interpretable in suitable  $\mathcal{M}_{\mathbf{s}, \bar{p}, n}$ , see §1 with, e.g.  $\bar{p} = \langle p_n : n < \omega \rangle$  with  $p_n$  going slowly to zero.

That is, fixing formulas  $\varphi_{\ell}(\bar{x}_{\ell}) \in \mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$  starting with  $\mathbb{Q}_{\bar{\mathbf{t}}}, \ell < k$  with no obvious connections we decide a priori that for a random enough  $\mathcal{G}_{n, \bar{p}} = ([n], R_{\ell}^{\mathcal{G}_{n, \bar{p}}})_{\ell < k}$  the structure  $(\mathcal{G}_{n, p}, R_{\ell}^{\mathcal{G}_{n, p}}, \dots)_{\ell < k}$  will look like  $R_{\ell}^{\mathcal{G}_{n, p}} = \{\bar{a} \in {}^{\ell}g(\bar{x}_{\ell})[n] : \mathcal{G}_{n, p} \models \varphi_{\ell}[\bar{a}]\}$ . The decision is the simplest one: look as if truth values of  $R_{\ell}^{\mathcal{G}_{n, p}}(\bar{a})$  were drawn independently, with probability  $p_n$ . This is an over simplification! We need a more involved such drawing, reflecting the original  $\bar{\varphi}_{\ell}$  to some extent, see below.

We may replace  $\mathcal{M}_{\mathbf{s}, \bar{p}, n}$  by using (for some irrational  $\alpha \in (0, 1)$ )  $\bar{p}_n = (p, p_n)$ , such that  $p_n = \frac{1}{n^{\alpha}}$ , except the original drawing of the graphs as in [ShSp:304]. We can also analyze  $G_{n, p, s, n^{-\alpha}}$  and use several pairs  $(r, \alpha)$  in the analysis (as long as the sets of  $\alpha$ 's is linearly independent over the rationals). Probably for some such version there is a more reasonably definable  $\mathbf{Q}_{\mathbf{K}}$  which immitate its behavior.

So in the proof we have two questions to address: fixing  $G = ([n], R_{\ell})_{\ell < k}$ , drawing the quantifiers, how  $(G, R_{\ell}^G, \dots)$  look like. Second, we need to consider all the  $G$ 's on  $[n]$ . For the first stage the main problems are: two definable derived graphs which are isomorphic.

We do some kind of elimination of quantifiers: essentially if  $\mathcal{M}_n$  is a  $\tau$ -structure ( $\tau$  relational and finite) drawn randomly according to the sequence  $\langle p_{\tau, R} : R \in \tau \rangle$  of fixed probabilities, applying  $\mathbf{Q}_{\bar{\mathbf{t}}}$  to some finitely many schemes  $\langle \mathbf{s}_0, \dots, \mathbf{s}_k \rangle$  of interpreting graphs, gives, i.e. a random  $\mathcal{M}'_n$  for  $\tau'$ -structures by expanding  $\mathcal{M}_n$  by  $R_{\ell} = \{\bar{c} : \ell g(\bar{c}) = \ell g(\bar{z}_{\mathbf{s}_{\ell}})\}$  and the graph  $H_{\mathbf{s}_{\ell}, \bar{c}}$  interpreted by  $\mathbf{s}_{\ell}$  for the parameter  $\bar{c}$  is in the class  $\mathbf{Q}_{\bar{\mathbf{t}}}\}$ .

Our use of vocabulary and structure deviates a little from the standard, but fits in the use in graph theory and is natural here. In graph theory the edge relation  $R$  is assume to be symmetric and irreflexive. So we use (say  $k_t$ -place predicate)  $R_t$  such that it is always irreflexive (fails for  $k_t$ -tuples with a repetition) and  $K_t$ -invariant, i.e. if  $\langle a_{\ell} : \ell < k_t \rangle$  satisfies it then so does  $\langle \bar{a}_{\pi(\ell)} : \ell < k_t \rangle$  for every  $\pi \in K_t$ . This is natural because when  $\bar{\varphi}(\bar{c})$  defines a graph  $H = H_{M, \bar{\varphi}, \bar{c}}$  in the structure  $M$  (e.g. a

graph) and we like to draw a truth value for “ $H \in \mathbf{K}_{\mathbf{t}}$ ”, a group of permutation of  $\ell g(\bar{c})$  is dictated by  $\bar{\varphi}$ .

Why the random auxiliary stucturs are better defined in a different way? Recall the truth value of “ $H \in \mathbf{K}_{\mathbf{t}}$ ” is chosen randomly, but if  $H$  is definable in the graph  $G$ , say is  $H_{G, \bar{\varphi}, \bar{c}}$ . The probability of “ $H \in \mathbf{K}_{\mathbf{t}}$ ” depends on  $H$ , and in natural cases, on  $|H|$  the number of nodes of  $H$ . But if  $\mathcal{M} = ([n], \dots, R_\ell^0, \dots)$  is random, the standard was to make the probability of  $\bar{c} \in R_\ell^G$  naturally depend on  $n$  and in many cases  $n \neq |H|$ .

We could have allowed using the quantifiers only on graph  $H$  definable in  $\mathcal{G}_{n,q}$  with not of nodes  $[n]$  but this seems to me quite undesirable. We restrict ourselves to the class of graphs - twice, we consider  $\mathcal{G}_{n,q}$  and the quantifier  $\mathbf{Q}_{\mathbf{t}}$  is on graphs. But in both cases this is not really needed.

We thank Simi Haber for raising again the problem and for some stimulating discussions.

## § 1. IDENTIFYING THE LOW GRAPHS

We like to add a quantifier  $\mathbf{Q}$  on finite graphs, which give a property of finite graphs respecting isomorphism (or a subset closed under automorphisms). The aim is that for e.g. the random graph  $\mathcal{G}_{n,p}$ , the 0-1 law holds for  $\mathbb{L}(\mathbf{Q})$ .

More specifically, we better make the quantifier trivial on too simple graphs, then for any fix finite set of formulas from  $\mathbb{L}(\mathbf{Q})$ , for random enough  $G_{n,p}$  the structure  $(G, \varphi^G(-))_{\varphi \in \Delta}$  is a random structure excluding the “problematic” graphs.

## § 1(A). Interpretation.

**Convention 1.1.** 1)  $\mathbf{h} : \mathbb{N} \rightarrow (0, \frac{1}{2})_{\mathbb{R}}$  goes to zero slowly enough, e.g.  $\mathbf{h}(n) = 1/\log_2 \log_2(n)$  for  $n > 16$  and  $= 1$  if  $n \leq 16$  which means:

- (a)  $\alpha \in (0, 1)_{\mathbb{R}} \Rightarrow 0 = \lim \langle \mathbf{g}(n)/n^\alpha : n < \omega \rangle$
- (b)  $\mathbf{g}$  is non-decreasing.

2)  $\mathbf{g} : \mathbb{N} \rightarrow \mathbb{R}_{\geq 2}$  be  $\mathbf{g}(n) = n^{\mathbf{h}(n)}$  be  $\mathbf{g}(n) = n^{\mathbf{h}(n)}$  hence  $\mathbf{g}(1+n) \geq 1$ ,  $\mathbf{g}$  go to infinity slowly enough.

*Notation 1.2.* 1) Let  $[n] = \{1, \dots, n\}$  or  $\{0, \dots, n-1\}$  if you prefer (serve as the universe of the  $n$ -th random graph).

2)  $\tau$  denotes a vocabulary (e.g.  $\tau = \tau_{\text{gr}}$  is the vocabulary of graphs; see Definition 1.3 below).

3) A  $\tau$ -model  $M$  is defined as usual.

4) For a formula  $\varphi = \varphi(\bar{x}, \bar{y})$ , model  $M$  and  $\bar{b} \in {}^{\ell g(\bar{y})}M$  let  $\varphi(M, \bar{b}) = \{\bar{a} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{a}, \bar{b}]\}$ .

**Definition 1.3.** 1) For a finite set  $I$  we say  $\mathbf{s}$  is an  $I$ -kind or a kind sequence (of a vocabulary) and write  $I_{\mathbf{s}} = I$  when:

- (a)  $\mathbf{s} = \langle (k_t, K_t) : t \in I \rangle = \langle (k_{\mathbf{s},t}, K_{\mathbf{s},t}) : t \in I \rangle$
- (b)  $k_t \in \mathbb{N}$
- (c)  $K_t$  is a group of permutations of  $\{0, \dots, k_t - 1\}$ .

1A) Let  $\mathbf{s}_{\text{gr}} = \mathbf{s}(\text{gr})$  be defined by (gr stands for graphs)  $I_{\mathbf{s}} = \{s_{\text{gr}}\}$ ,  $s_{\text{gr}}$  fix, e.g.  $0, k_{\mathbf{s},s_0} = 2, K_{\mathbf{s},s_0} = \text{sym}(2)$ , the group of permutations of  $\{0, 1\}$ .

2) For  $\mathbf{s}$  an  $I$ -kind we define:

- (a) the  $\mathbf{s}$ -vocabulary  $\tau_{\mathbf{s}}$  is  $\{R_t : t \in I\}$ ,  $R_t$  a  $k_{\mathbf{s},t}$ -place predicate
- (b) an  $\mathbf{s}$ -structure is  $M = (|M|, R_t^M)_{t \in I}$  such that:
  - ( $\alpha$ )  $R_t^M$  is a  $k_t$ -place relation on  $|M|$
  - ( $\beta$ )  $R_t^M$  is  $K_t$ -invariant, i.e. if  $\langle a_\ell : \ell < k_t \rangle \in R_t^M \wedge \bar{b} \in \bar{a}/E_{K_t} \Rightarrow \bar{b} \in R_t^M$  where  $\bar{a}/E_{K_t} = \{\langle a_{\pi(\ell)} : \ell < n_t \rangle : \pi \in K_t\}$ ; let  $E_{\mathbf{s},t} = E_{K_t}$
  - ( $\gamma$ )  $R_t^M$  is irreflexive, i.e.  $\bar{a} \in R_t^M \Rightarrow \bar{a}$  with no repetitions.
- (c)  $\mathbf{M}_{\mathbf{s}} = \cup \{\mathbf{M}_{s,m} : m \in \mathbb{N}\}$  where  $\mathbf{M}_{s,m} = \{M : M \text{ an } \mathbf{s}\text{-structure with set of elements } [m]\}$ .

- 3) For an  $I$ -kind  $\mathbf{s}$  let  $\mathbf{P}_\mathbf{s}^1$  be the set of  $\bar{p} = \langle p_{t,n} : t \in I, n \in \mathbb{N} \rangle, p_{t,n} \in (0,1)_{\mathbb{R}}$ . We define the  $(\mathbf{s}, \bar{p})$ -random structure on  $[n]$ ,  $\mathcal{M} = \mathcal{M}_{\mathbf{s}, \bar{p}, n}$  as follows: for  $t \in I$  and  $\bar{a} \in {}^{k_t}([n])$  with no repetitions we draw a truth value for  $\bar{a} \in R_t^\mathcal{M}$  with probability  $p_{t,n}$ , but demanding we have the same result for  $\bar{a}', \bar{a}''$  when they are  $E_{\mathbf{s}, t}$ -equivalent and independent of the rest.
- 3A) Let  $\mathbf{P}_\mathbf{s}^0$  for  $\mathbf{s}$  as above be the set of  $\bar{p} \in \mathbf{P}_\mathbf{s}^1$  such that  $g \in I_\mathbf{s} \wedge n \in \mathbb{N} \Rightarrow p_{t,n} = p_{t,0}$ , so we may write  $p_t$  instead of  $p_{t,0}$ .
- 4) Let  $\mathbf{P}_\mathbf{s}^2$  be the set of  $\bar{p} \in \mathbf{P}_\mathbf{s}^1$  such that for some  $\bar{q} \in \mathbf{P}_\mathbf{s}^0$  and partition  $I_0, I_1$  of  $I$ , we have  $p_{t,n}$  in  $q_t$  if  $n \in I_0$  and is  $q_0/g(n)$  if  $t \in I_1$ ; we denote  $\bar{p}$  by  $\bar{p}_{\bar{q}, I_0}$ .
- 5) Let  $\mu_{\mathbf{s}, \bar{p}, n}$  be the distribution (= probability space) on  $\mathbf{M}_{\mathbf{s}, n}$  corresponding to drawing the truth value of  $R_t(\bar{a})$  really of  $\langle \mathbf{R}_t(\bar{a}') : \bar{a}' \in {}^{\bar{a}}E_{\mathbf{s}, t}$  for a sequence  $\bar{a}$  with no repetitions of length  $k_{\mathbf{s}, t}$  with probability  $p_{n,t}$ , independently of the other choices.
- 6) Let  $\mathcal{M}_{\mathbf{s}, \bar{p}, n}$  be the random variable for the finite probability space  $(\mathbf{M}_{\mathbf{s}, n}, \mu_{\mathbf{s}, \bar{p}, n})$ .
- 7) If  $\mathbf{s} = \mathbf{s}_{\text{gr}, q}$  let  $\mathcal{G}_{n,q} = \mathcal{M}_{\mathbf{s}_{\text{gr}}, \bar{p}_{\text{gr}, q}, n}$  and  $\mu_{\text{gr}, q, n} = \mu_{\mathbf{s}_{\text{gr}}, \bar{p}_{\text{gr}, q}, n}$ .

Recall

- Fact 1.4.** 1)  $\mathbf{P}_\mathbf{s}^0 \subseteq \mathbf{P}_\mathbf{s}^2 \subseteq \mathbf{P}_\mathbf{s}^1$ .  
 2) For every  $\bar{p} \in \mathbf{P}_\mathbf{s}^2$ ,  $\mathcal{M}_{\mathbf{s}, \bar{p}, n}$  satisfies the 0-1 law for first order logic and the limit theory  $T_{\mathbf{s}, \bar{p}}$  has elimination of quantifiers, really is  $T_\mathbf{s}$ , i.e. does not depend on  $\bar{p}$  and  $\mathbf{g}$  and  $\mathbf{h}$  (as long as they are as in 1.1(2)).  
 3)  $\mathbf{M}_{\mathbf{s}_{\text{gr}}, n}$  is the set of graphs with set of nodes  $[n]$ .  
 4) For any  $q \in (0,1)_{\mathbb{R}}$  defining  $\bar{p}_{\mathbf{s}_{\text{gr}}, q}$  by  $p_{\mathbf{s}_{\text{gr}}, \mathbf{s}_{\text{gr}}, n} = q$  then  $\mathcal{G}_{n,q}, \mathcal{G}_{\mathbf{s}_{\text{gr}}, \bar{p}_{\text{gr}, q}, n}$  are the same.

*Proof.* Should be clear. □<sub>1.4</sub>

*Remark 1.5.* We first concentrate on one application of the quantifier.

We are interested in interpreting graphs. We give the most general case. Note we intend the quantifier to be a property of graphs. So we have to think of an interpretation of a graph. In such general interpretations using quantifier free formulas the elements may be only: a set of elements definable by a formula  $\varphi(x, \bar{a})$ ,  $\bar{a}$  is a sequence of parameters or more generally such a set of  $k$ -tuples, maybe modulo suitable  $E_K$ , or even a finite union of such. For each pair of the nodes (fixing from where in the union they come) we define when it is an edge by a quantifier free formula. So below  $\bar{z}$  are parameters,  $\mathbf{i}(\bar{\varphi})$  number of “kinds”, ways to define a node;  $\varphi_{0,i}$  describes the  $i$ -th kind,  $\varphi_2(\bar{z})$  describes the relevant parameters,  $\varphi_{1,i,j}$  describes the edges between a node of the  $i$ -th kind and a node of the  $j$ -th kind.

**Definition 1.6.** 1) For  $\mathbf{s}$  an  $I$ -kind, we say  $\bar{\varphi}$  is a  $\mathbf{s}$ -scheme (of a graph interpretation in  $\mathbf{s}$ -structures) when it consists of:

- (a)  $\langle \varphi_{0,i}(\bar{x}_i, \bar{z}), \varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z}), \varphi_2(\bar{z}) : i, j < \mathbf{i}(\bar{\varphi}) \rangle$  such that
  - <sub>1</sub>  $\ell g(\bar{x}'_j) = \ell g(\bar{x}_j)$ , it is possibly zero and
  - <sub>2</sub>  $\langle \bar{x}_i, \bar{x}'_i : i < \mathbf{i}(\bar{\varphi}) \rangle$  are pairwise disjoint, each with no repetitions
  - <sub>3</sub>  $\mathbf{i}(\mathbf{s})$  is a non-zero natural number
- (b)  $\varphi_{0,i}, \varphi_{1,i,j}, \varphi_2$  are quantifier free formulas in  $\mathbb{L}(\tau_\mathbf{s})$
- (c)  $K_i$  is a group of permutations of  $\{0, \dots, \ell g(\bar{x}_i) - 1\}$  not related to  $K_{\mathbf{s}, t}(t \in I)!$

- (d)  $\varphi_{0,i}(\bar{x}_i, \bar{z})$  is invariant under permuting  $\bar{x}_i$  by any  $\pi \in K_i$  and  $\varphi_2(\bar{z}) \vdash (\exists \bar{x}_i) \varphi_{1,1}(\bar{x}_i, \bar{z})$
- (e)  $\varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z})$  is invariant under permuting  $\bar{x}_i, \bar{x}'_j$  by  $\pi \in K_i, \pi' \in K_j$  respectively, and  $\vdash \varphi_{1,i,j}(\bar{x}_i, \bar{x}'_j, \bar{z}) \equiv \varphi_{1,j,i}(\bar{x}'_j, \bar{x}_i, \bar{z})$  and  $\vdash \neg \varphi_{1,i,i}(\bar{x}_i, \bar{x}_i, \bar{z})$
- (f) if  $i \in \{0, 1, 2\}$  and  $M$  is a  $\tau_s$ -structure and  $G \models \varphi_{0,i}[\bar{a}, \bar{c}]$ , so  $\ell g(\bar{c}) = \ell g(\bar{z})$  then  $\bar{a} \hat{=} \bar{c}$  is with no repetitions.
- 1A) So if we have  $\bar{\varphi} = \bar{\varphi}'$  then  $\varphi'_{0,i} = \varphi_{0,i}$ , etc. and we may write  $\bar{z}_{\bar{\varphi}}, \bar{x}_{\bar{\varphi},1,i}, \bar{x}'_{\bar{\varphi},1,1}$ .
- 2) If  $s$  and  $\bar{\varphi}$  are as above,  $M = \mathcal{M}_{s,\bar{p},n}$  and  $\bar{c} \in {}^{\ell g(\bar{z})}M$  satisfies  $M \models \varphi_2[\bar{c}]$  then  $H = H_{\bar{\varphi}, \mathcal{M}, \bar{c}}$  is a graph, where
- ( $\alpha$ ) the set of nodes is  $\{(i, \bar{a}/E_{K_i}^M) : M \models \varphi_{0,i}[\bar{a}, \bar{c}] \text{ for some } i < \mathbf{i}(\bar{\varphi}) \text{ and } \bar{a} \in {}^{\ell g(\bar{x}_i)}M\}$
- ( $\beta$ )  $\{(i, \bar{a}/E_{K_i}^M), (j, \bar{b}/E_{K_j}^M)\}$  is an edge iff  $M \models \varphi_{1,i,j}[\bar{a}, \bar{b}, \bar{c}]$ .
- 3) Let  $k_s(\bar{\varphi}) = \max(\{\ell g(\bar{x}_i) : i < \mathbf{i}(\bar{\varphi})\} \cup \{\ell g(\bar{z})\})$  let  $k_{s,i}(\bar{\varphi}) = \ell g(\bar{x}_i), k_{s,*}(\bar{\varphi}) = \max\{\ell g(\bar{x}_i) : i < \mathbf{i}(\bar{\varphi})\}$ .

**Observation 1.7.** In Definition 1.6,  $H_{\bar{\varphi}, M, \bar{c}}$  is indeed a graph except possibly being empty and is finite when  $M$  is finite  $\tau_s$ -model.

*Proof.* Read Definition 1.6(1). □<sub>1.7</sub>

**Observation 1.8.** 1) Let  $s$  be an  $I$ -kind and  $\bar{\varphi}$  is a  $s$ -scheme. The following are equivalent:

- (a) for every  $\bar{p} \in \mathbf{P}_s^2$  and random enough  $\mathcal{M} = \mathcal{M}_{s,n}$  we have  $\varphi_{s_2}(\mathcal{M}) \neq \emptyset$
- (b) for some  $\bar{p} \in \mathbf{P}_s^2$ , e.g.  $\bar{p} \in \mathbf{P}_s^1$  we have  $1 > \limsup_n \text{Prob}(\varphi_2(\mathcal{M}_{s,p,n}) \neq \emptyset)$ .
- 2) Similarly replacing  $\varphi_2(\mathcal{M}) \neq \emptyset$  by  $H_{\bar{\varphi}, \mathcal{M}, \bar{c}} \models \psi$ .

**Definition 1.9.** 1) We call an  $s$ -scheme  $\bar{\varphi}$  trivial when for each  $i < \mathbf{i}(\varphi)$  we have  $\ell g(\bar{x}_i) = 0$ .

2) We call a  $s$ -scheme  $\bar{\varphi}$  degenerated when the conditions of 1.8 fail.

3) We say the  $s$ -scheme  $\bar{\varphi}$  is 1-weak when at least one of the following holds:

- (a)  $s$  is trivial, i.e.  $\ell g(\bar{x}_i) = 0$  for every  $i < \mathbf{i}(\varphi)$
- (b)  $s$  is degenerated
- (c) for some truth value  $\mathbf{t}$  and  $i_1, i_2 < \mathbf{i}(\varphi)$  satisfying  $\ell g(\bar{x}_{i_1}), \ell g(\bar{x}_{i_2}) \geq 1$  and  $v_1 \subsetneq \ell g(\bar{x}_{i_1}), v_2 \subsetneq \ell g(\bar{x}_{i_2})$  we have
- some (equivalent only)  $\bar{p} \in \mathbf{P}_s^2$  for random enough  $\mathcal{M} = \mathcal{M}_{s,n}$ , for some  $\bar{c} \in \varphi_2(\mathcal{M}), \bar{\mathbf{a}}_\ell^* \in \varphi_{1,i_\ell}(\mathcal{M}, \bar{c})$  for  $\ell = 1, 2$  we have
  - if  $\bar{a}_\ell \in \varphi_{1,i_\ell}(\mathcal{M}, \bar{c})$  and  $\bar{a}_\ell \upharpoonright v_\ell = \bar{a}_\ell^* \upharpoonright v_\ell$  for  $\ell = 1, 2$  and  $\text{rang}(\bar{a}_1) \cap \text{rang}(\bar{a}_2) \subseteq \text{rang}(\bar{a}_1^* \upharpoonright v_1) \cup \text{rang}(\bar{a}_2^* \upharpoonright v_2)$  and  $\text{rang}(\bar{a}_\ell)$  is disjoint to  $\text{rang}(\bar{a}_1^*) \cup \text{rang}(\bar{a}_2^*)$  then  $\mathcal{M} \models \varphi_{i_1, i_2}[\bar{a}_1, \bar{a}_2, \bar{c}]^{\text{if}(\mathbf{t})}$ .

4) We say the  $s$ -scheme  $\bar{\varphi}$  is 2-weak when at least one of the following holds:

- (a), (b) as above
- (c) for some  $i < \mathbf{i}(\varphi), \ell g(\bar{x}_i) \geq 2$

- (d) for some  $i_1, i_2 < i(\varphi)$  with  $\ell g(\bar{x}_{i_2}) = 1 = \ell g(\bar{x}_{i_1})$  and  $\bar{p} \in \mathbf{P}_s^2$  and random enough  $\mathcal{M} = \mathcal{M}_{s, \bar{\varphi}, n}$  and  $\bar{c} \in \varphi_2(\mathcal{M})$  there are  $a_2^t \in \varphi_{1, i_1}(\mathcal{M}), a_2^t \in \varphi_{1, i_2}(\mathcal{M})$  for  $t = 0, 1$  such that  $a_1^t \neq a_2^t$ ; and  $H_{s, \mathcal{M}, \bar{c}} \models "a_1^t R a_2^t \text{ iff } t = 1"$ .

**Claim 1.10.** 1) For any  $k$ , if  $\mathcal{M} = \mathcal{M}_{s, \bar{p}, n}$  is random enough for  $k$  and  $\bar{c} \in {}^{k \geq M}$ , and there is an interpretation of a graph using as parameter  $\bar{c}$  of a graph  $H$  in  $\mathcal{M}$  using  $(\leq k)$ -tuples (in the widest sense - the elements can be equivalent classes of suitable equivalence relations on set of tuples satisfying a formula) by formulas of length  $\leq k$  then there is a  $s$ -scheme  $\bar{\varphi}$  such that  $H = H_{\bar{\varphi}, \mathcal{M}, \bar{c}}$  and  $k(\bar{\varphi}) \leq k$ .

2) In fact  $\bar{\varphi}$  depends just on the interpretation and the quantifier free type of  $\bar{c}$  in  $\mathcal{M}$ , not on  $\mathcal{M}$  (and even  $n$ ).

3) Given  $s$  and  $k$  there only finitely many  $\bar{\varphi}$ 's scheme  $\bar{\varphi}$  as above.

*Proof.* Obvious. □<sub>2.4</sub>

### § 1(B). Simple Random Graph.

We have began thinking that the behaviour of  $\mathcal{G}_{n, q}$  expanded by some formulas in the expanded logic will be like  $\mathcal{G}_{s, \bar{p}}, \bar{p} \in \mathbf{P}_s^2$ , but we need a relative.

**Definition 1.11.** Let  $\mathbf{U}$  be the set of objects  $\mathbf{u}$  consisting of the following (we may add subscript  $\mathbf{q}$ ):

- (a)  $\bar{s} = \langle s_\ell : \ell \leq \ell(\mathbf{q}) \rangle$
- (b)  $s_\ell$  is a kind sequence
- (c)  $s_0 = s_{gr}$ , the graph kind sequence, see 1.3(1A)
- (d)  $s_\ell \subseteq s_{\ell+1}$ , i.e.  $I_{s_\ell} \subseteq I_{s_{\ell+1}}$  and  $t \in I_{s_\ell} \Rightarrow (k_{s_\ell, t}, K_{s_\ell, t}) = (k_{s_{\ell+1}, t}, K_{s_{\ell+1}, t})$
- (e) notation so we may write  $(k_{\mathbf{u}, t}, K_{\mathbf{u}, t})$  for  $t \in I_{s_{\ell(\mathbf{u})}}$  and  $I_{\mathbf{q}} = I_{s_{\ell(\mathbf{u})}}$
- (f) for  $t \in I_{s_{\ell+1}} \setminus I_{s_\ell}$  we have  $\mathbf{i}(t), K_t$  a group of permutation of  $\ell g(\bar{z}_t), \varphi_t(\bar{z}_t)$  a complete quantifier formulation in  $\mathbb{L}(\tau_{s_\ell})$  with  $k_t$  variables (so saying they are pairwise distinct and is  $K_t$ -invariant,  $\mathbf{i}(t) \in \mathbb{N}$  and  $\psi_{t, i}(\bar{y}_{t, i}, \bar{z}_t)$  also complete quantifier free formulas (not necessarily distinct) such that  $\psi_{t, i}(\bar{y}_{t, i}, \bar{z}_t) \vdash \varphi_t(\bar{z}_t), \bar{y}_{t, i}$  of length  $b_{t, i}, K_{t, i}$  a group of permutation of  $\ell g(\bar{y}_{t, i})$  such that  $\psi_{t, i}$  is  $K_{t, i}$ -invariant. In the case  $\iota = 2$  the  $\bar{y}_{t, i}$  is a singleton so we shall write  $\psi(y, \bar{z}_{t, i})$
- (g)  $q = q_{\mathbf{u}} \in (0, 1)_{\mathbb{R}}$ .

**Definition 1.12.** For  $\mathbf{u} \in \mathbf{U}$  we define a random set  $\mathcal{M}_{\mathbf{u}, n}$ , i.e. a 0-1 content, as follows.

For a given  $n$ ,  $\mathcal{M}_{\mathbf{u}, n}$  is by drawing  $\mathcal{M}_{\mathbf{u}, n, \ell} \in \mathbf{M}_{s_{\mathbf{u}, \ell}, n}$  by induction on  $\ell \leq \ell(\mathbf{u})$  and in the end  $\mathcal{M}_{\mathbf{u}, n} = M_{\mathbf{u}, n, \ell(\mathbf{u})}$ .

Now

- (a) if  $\ell = 0$ ,  $\mathcal{M}_{\mathbf{u}, n, \ell}$  is  $\mathbf{M}_{q(\mathbf{u}), n}$ , i.e. the random graph on  $n$  with edge probability  $q$
- (b) if  $\ell < \ell(\mathbf{u})$  and  $\mathcal{M}_{\mathbf{u}, n, \ell}$  has been drawn and  $t \in I_{s_{\ell+1}} \setminus I_{s_\ell}$ , we draw  $R_t(\mathcal{M}_{s_{\ell+1}})$  as follows:
  - (α) if  $\bar{c} \in \varphi_t(M)$  we draw the truth value of  $\bar{c} \in R_t(\mathcal{M}_{s_{\ell+1}, n})$  with probability  $\mathbf{h}(\sum_{i < i(q)} |\psi(\mathcal{M}_{s_{\ell+1}, n}, \bar{c})| / |K_{t, i}|)$



( $\beta$ ) if  $\bar{c}$  is a sequence of length  $k_t$  but  $\notin \varphi_t(M)$  then  $\bar{c} \notin R_t(\mathcal{M}_{\mathbf{s}_{\ell+1}, t})$ .

**Claim 1.13.** For  $\mathbf{u} \in \mathbf{U}$ ,  $\mathcal{M}_{\mathbf{u}, n}$  like  $\mathcal{M}_{\mathbf{s}_{\mathbf{q}}, \bar{p}}$  for any  $\bar{p} \in \mathbf{P}_{\mathbf{s}_{\mathbf{q}}}^2$  (and  $M_{\mathbf{u}, n, \ell}$  like  $\mathcal{M}_{\mathbf{s}_{\mathbf{q}}, \ell, \bar{p}}$ ), in particular, satisfying the zero one law:

(\*) for any  $k_1$  for some  $k_2$ , for any random enough  $\mathcal{M}_{\mathbf{u}, n}$  we have:

- if  $\varphi(\bar{x}), \psi(\bar{y}, \bar{z})$  are complete  $\mathbb{L}(\tau_{\mathbf{s}_{\mathbf{u}}})$ -formulas such that  $\psi(\bar{y}, \bar{z}) \vdash \varphi(\bar{z})$  (so they respect the  $K_{\mathbf{u}, t}$ 's!, see yyy) and  $\ell g(\bar{y}) + \ell g(\bar{x}) \leq k_1$  and  $\bar{c} \in \text{varphi}(\mathcal{M}_{\mathbf{u}, n})$  and  $k_{t, i} \geq 1$  then the number of members of  $\psi_{t, i}(M_{\mathbf{u}, n}, \bar{c})$  is similar to  $\binom{n^{\ell g(\bar{y})}}{k_t} \cdot \frac{k_t}{(K_t)}$  fully
- at most<sup>1</sup>  $\binom{n^{\ell g(\bar{y}_{t, i})}}{k_{t, i}} \cdot \frac{k_{t, i}!}{|K_{t, i}|} \cdot (1 - \frac{1}{k_2})$
- at least  $\binom{n^{k_{t, i}}}{k_t} \cdot \frac{k_{t, i}!}{|K_{t, i}|} \cdot \mathbf{h}(n) - k_2$
- if  $\iota = 2$ , then  $k_{t, i} = 1$ , so this is simpler.

*Proof.* should be clear. □<sub>S</sub>

### § 1(C). Low/High Graphs.

Schemes  $\bar{\varphi}$  may be such that, e.g. the bi-partite graph with the  $i$ -th kind and the  $j$ -th kind is trivial. Those cases are “undesirable” for us and we shall try to discard them.

**Definition 1.14.** 1) We say a finite graph  $H$  is  $\mathbf{h} - 1$ -low (recall  $\mathbf{h}$  is from 1.1 so can be omitted) when there are no disjoint  $A, B \subseteq H$  and  $\iota < 2$  such that (letting  $n = |H|$ )

- (a)  $|A|, |B| \geq |H|^{\mathbf{h}(n)}$
- (b) if  $a \in A$  and  $b \in B$  then  $(a, b)$  is an edge of  $H$  iff  $\iota = 1$ .

2) We say that a finite graph  $H$  is  $\mathbf{h} - 2$ -low when letting  $n = |H|, m = \lfloor \log(\log(n)) \rfloor$  there are no  $\bar{a}, \bar{b}, M, \mathbf{c}$  such that:

- (a)  $\bar{a} = \langle a_\ell : \ell < m \rangle$
- (b)  $\bar{b} = \langle b_{\ell, k} : \ell < k \leq m \rangle$
- (c)  $\bar{a} \hat{\ } \bar{b}$  is a sequence of nodes of  $H$
- (d)  $\mathbf{c}_1$  is a function from  $\{(\ell_1, \ell_2, m_2) : \ell_1, \ell_2, m \leq m\}$  to  $\{0, \dots, \lfloor \log(\log(m)) \rfloor\}$
- (e)  $\mathbf{i}_2$  is a function from  $\{(\ell, k) : \ell \leq m, k < m\}$  into  $\{0, \dots, \lfloor \log(\log(m)) \rfloor\}$
- (f) if  $\ell' < k' \leq m$  and  $j' < m$  and  $\ell'' < k'' \leq m, f'' \leq n$  and  $\mathbf{c}_1(\ell', k', j') = \mathbf{c}(\ell'', k''), \mathbf{c}_2(\ell', j') = \mathbf{c}_1(\ell', j'')$  and  $\mathbf{c}_2(k', j') = \mathbf{c}_1(k'', j'')$  then  $(b_{\ell', k'}, a_{j'})$  is an edge of  $H$  iff  $(b_{\ell'', k''), a_{j''})$  is an edge of  $H$ .

3) In part (1) and (2),  $\mathbf{h} - \iota$ -high means the negation of  $\mathbf{h} - \iota$ -low.

**Claim 1.15.** Assume  $\mathbf{s}$  is an  $I$ -kind, (see Definition 1.3) and  $\bar{\varphi}$  is a non-degenerated  $\mathbf{s}$ -scheme (see Definition 1.6, 1.9(2))

---

<sup>1</sup>we could have allowed, e.g. when  $k_t = 1$  to be near to 1 though not too closely, but if we shall use a quantifier  $\mathbf{Q}$  such that  $\ll \frac{1}{2}$  of the structures are in it

(A) the following are equivalent:

- (α)  $\bar{\varphi}$  is trivial
- (β) if  $\bar{p} \in \mathbf{P}_s^2$  then for random enough  $\mathcal{M} = \mathcal{M}_{s,n,\bar{p}}$  and  $\bar{c} \in \varphi_2(\mathcal{M})$  the graph  $H_{\bar{\varphi},\mathcal{M},\bar{c}}$  has  $\leq \mathbf{i}(\bar{\varphi})(k(\bar{\varphi})!)$  nodes
- (γ) if  $\varepsilon > 0$  and  $\bar{p} \in \mathbf{P}_s^2$  then  $1 > \limsup_n \text{Prob}(\text{letting } \mathcal{M} = \mathcal{M}_{s,\bar{p},n}, \text{ for some } \bar{c} \in \varphi_2(\mathcal{M}) \text{ the graph } H_{\bar{\varphi},\mathcal{M},\bar{c}} \text{ has } \leq n^{1-\varepsilon} \text{ nodes})$

(B) the following are equivalent:

- (α)  $\bar{\varphi}$  is 1-weak
- (β) if  $\bar{p} \in \mathbf{P}_s^2$  then for every random enough  $\mathcal{M} = \mathcal{M}_{s,n,\bar{p}}$  and  $\bar{c} \in \varphi_2(\mathcal{M})$  the graph  $H_{\bar{\varphi},\mathcal{M},\bar{c}}$  is 1-low
- (γ) if  $\varepsilon > 0$  and  $\bar{p} \in \mathbf{P}_s^2$  then  $1 > \limsup_n \text{Prob}(\text{letting } \mathcal{M} = \mathcal{M}_{s,\bar{p},n}, \text{ for some } \bar{c} \in \varphi_2(\mathcal{M}) \text{ the graph } H_{\bar{\varphi},\mathcal{M},\bar{c}} \text{ is 1-low})$

(C) Like (B), replacing 1-weak, 1-low by 2-weak, 2-low respectively.

*Proof.* Clause (A):

Trivially  $(A)(\alpha) \Rightarrow (A)(\beta)$  and  $(A)(\beta) \Rightarrow (A)(\gamma)$

So it suffices to assume  $\bar{\varphi}$  is non-trivial,  $\bar{p} \in \mathbf{P}_s^2$  and let  $\varepsilon > 0$  be small enough and prove that for every random enough  $\mathcal{M} = \mathcal{M}_{s,\bar{p},n}$  and  $\bar{c} \in \varphi_2(\mathcal{M})$  the graph  $H_{\bar{\varphi},\mathcal{M},\bar{c}}$  has  $\geq \varepsilon n$  nodes.

Let  $i < \mathbf{i}(\bar{\varphi})$  be such that  $k_i = \ell g(\bar{x}_i) > 0$ , so for  $n$  large enough and  $\bar{c} \subseteq [n]$  of length  $\ell g(\bar{z})$  let  $S_{n,\bar{c}} = \{\bar{a} : \bar{a} \text{ is a sequence of length } \ell g(\bar{x}_i) \text{ with no repetition of members of } [n] \text{ not from } \bar{c}\}$ . For every  $\bar{a} \in S_{n,\bar{c}}$ ,  $\text{Prob}(\mathcal{M}_{\bar{\varphi},n,\bar{p}} \models \text{"if } \varphi_2(\bar{c}) \text{ then } \varphi_1(\bar{a}, \bar{c})\text{"})$  is the same for every  $\bar{a} \in S_{n,\bar{c}}$  and is of the form  $r/g(n)^m$  for some  $r_1 \in (0, 1)_{\mathbb{R}}, m \in \mathbb{N} \setminus \{0\}$  not depending on  $n$ . Clearly the probability of “no such  $\bar{a}$ ” is  $\leq 2^{r(2)^n}$  for some  $r(2) \in (0, 1)_{\mathbb{R}}$ . Hence the probability of failure for some  $\bar{c}$  is  $\leq 2^{r(2)^{m/2}}$ , so we can ignore it.

Clause (B):

First why  $(B)(\alpha) \Rightarrow (B)(\beta)$ ?

Considering Definition 1.9(3), if clause (a) there holds, i.e.  $s$  is trivial then by Clause (A) here we are done. Next  $(s, \bar{\varphi})$  cannot satisfy clause (b) of Definition 1.9 because in the present claim we are assuming  $\bar{\varphi}$  is non-degenerated. So assume clause (c) of 1.9(3) holds as exemplified by  $i_1, i_2, \mathbf{i}(\bar{\varphi}), v_1, v_2$  and truth value  $\mathbf{t}$ , i.e.  $\ell g(x_i), \ell g(\bar{x}_j) > 0$ , etc. So assume  $n$  is large enough and  $\mathcal{M} = M, \bar{c} \subseteq [n]$  has length  $\ell g(\bar{z}_{\bar{\varphi}})$ .

Let  $A_\ell = \{\bar{a} : \bar{a} \subseteq [n] \text{ is of length } \ell g(\bar{x}_{i_\ell}) \text{ for } \ell = 1, 2 \text{ with no repetition and is disjoint to } \bar{c}\}$ . Choose  $\bar{a}_\ell^* \in A_\ell$ . So the event  $\mathcal{E}_{\bar{c}} = “(\bar{c} \hat{\wedge} \bar{a}_1^* \hat{\wedge} \bar{a}_2^*) \text{ is as in 1.9(3)}”$  has probability  $\geq r_1(\mathbf{g}(n))^{k(1)}$  for some  $r_1 \in (0, 1)_{\mathbb{R}}, k \in \mathbb{N} \setminus \{0\}$  not depending on  $n$  (and  $\bar{c}$ ). Fixing  $(\bar{c}, \bar{a}_1^*, \bar{a}_2^*)$  let  $C_i \subseteq (n^1 \setminus \text{rang}(\bar{c} \hat{\wedge} \bar{a}_1^* \hat{\wedge} \bar{a}_2^*), |C_i| \geq (n - |\ell g(\bar{z} \hat{\wedge} \bar{x}_{i_1} \hat{\wedge} \bar{x}_{i_1}) - \frac{1}{2}|$  for  $\ell = 1, 2$  and  $C_1 \cap C_2 = \emptyset$ . Let  $A'_\ell = \{\bar{a} \in A_\ell : \text{Rang}(\bar{a}) \subseteq C_\ell\}$ .

Easily the probability  $\mathcal{E}_2 = \mathcal{E}_{\bar{c}, \bar{a}_1^*, \bar{a}_2^*}^r$  of the following event is  $\geq 1 - 2^{-r(2)^n}$  where

$$(*) \mathcal{E}_2 \text{ means if } \mathcal{M} \models \varphi_2[\bar{c}] \wedge \varphi_{1,i_1}[\bar{a}_1^*] \wedge \varphi_{1,i_2}[\bar{a}_2^*] \text{ then } \bar{a}_\ell \in A'_\ell : \mathcal{M} \models \varphi[\bar{a}_{\ell,m}]] \geq n^{|u(\ell)|/2} \text{ for } \ell = 1, 2.$$

Clearly  $\mathbf{t}$  and  $A_{\mathcal{M},\ell}^* = \{\bar{a}/E_{K_{s,i,\ell}} : \bar{a} \in A'_\ell \text{ and } \mathcal{M} \models \varphi_{0,i_\ell}[a_{n,m}, \bar{c}]\}$  for  $\ell = 1, 2$  exemplifies  $H_{\bar{\varphi},\mathcal{M},\bar{c}}$  is low. As the number of  $\bar{c}, \bar{a}_1^*, \bar{a}_2^*$  is polynomial we can finish.

Second, why  $(B)(\beta) \Rightarrow (B)(\gamma)$ :

Read the clauses and Definition of 1.14.

Third,  $\neg(B)(\alpha) \Rightarrow \neg(B)(\gamma)$ : This suffices

Why this holds? Let  $\mathcal{M} = \mathcal{M}_{\mathbf{s}, \bar{p}, n}$  be random enough,  $\bar{c}_2 \in \varphi_2(\mathcal{M})$  and  $A_1, A_2 \subseteq H = H_{\bar{\varphi}, \mathcal{M}, \bar{c}}$  witness  $H$  is low, so  $|A_\ell| \geq n^{\mathbf{h}(n)}$ . So  $n_1^* = \min\{|A_1^*|; |A_2^*|\} \geq m^{\mathbf{h}(n)}$ .

Clearly for each  $\ell \in \{1, 2\}$  for some  $i(\ell) < \mathbf{i}(\bar{\varphi})$  we have

$$|\bar{a}/K_{\mathbf{s}, i, \ell} \in A_\ell : \bar{a} \in \varphi_{1, i}(\mathcal{M}, \bar{c})| \geq |A_\ell|/\mathbf{i}(\bar{\varphi}) = n_2^* \geq n^{\mathbf{h}(n)}/\mathbf{i}(\bar{\varphi}).$$

So for  $\ell = 1, 2$  we can find  $\langle \bar{a}_{\ell, m} : m < n_3^* = \sqrt{n_2^*}/k(\bar{\varphi})$  and partition  $v_\ell, u_\ell$  of  $\ell g(\bar{x}_\ell)$  such that:

- (\*) (a)  $\bar{a}_{\ell, m} \upharpoonright v_c = a_\ell^*$
- (b)  $\text{Rang}(\bar{a}_{\ell, m, r} \upharpoonright u_\ell) \cap \text{Rang}(\bar{a}_{\ell, m(2)}) = \emptyset$  for  $m_1 < m_2 < n_3^*(\bar{\varphi})$
- (c)  $\text{Rang}(\bar{a}_{\ell, m, r} \upharpoonright u_1), \text{Rang}(\bar{a}_{2, m(r)} \upharpoonright u_2), \bar{a}_1^* \hat{\wedge} \bar{a}_2^*$  are pairwise disjoint for  $m(1), m(2) < n_3^*$ .

We draw  $\mathcal{M} \upharpoonright (\bar{c} \hat{\wedge} \bar{a}_{\ell, m})$  for every  $\ell \in \{n\}$  and  $m < n_3^*$  we get  $\mathcal{M}'$ . So ignoring events of very low probability ( $\leq (\frac{1}{2})^{rn}$  for fix  $r$ )

- (\*) (a)  $w_\ell := \{m < n_3^* : (M' \upharpoonright \bar{c} \hat{\wedge} \bar{a}_{\ell, m}) \models \varphi_{1, i(\ell)}[\bar{a}_{\ell, m}, \bar{c}]\}$  has  $\geq n_4^* := \sqrt{n_3^*}$  members
- (b) renaming  $w_\ell = n_4^*$ .

So  $n_3^* = n^\varepsilon$  for  $\varepsilon$  small enough but let  $Y = \{\langle \bar{a}_{\ell, m} : \ell \in \{1, 2\}, m \in w_\ell : \bar{a}_{\ell, m} \text{ as above} \rangle\}$ .

So  $|Y| \sim \binom{n^{k(i(1))}}{n^\varepsilon} \cdot \binom{n^{k(i(2))}}{n^\varepsilon} \sim n^{(k, i(1)) + k(i(2)) \cdot n^\varepsilon} = e^{-k(\bar{\varphi}), n^\varepsilon \log(n)}$ .

For each  $\mathbf{a} \in Y(*)$  assume  $\mathcal{M} \upharpoonright \bar{a}_{\ell, m} \hat{\wedge} \bar{c} \models \varphi_{1, i(\ell)}(\bar{a}_{\ell, n}, \bar{c})$  for  $\ell \in \{1, 2\}, m < n_4^*$  is  $\sim (1 - q)^{n^{2\varepsilon}} = (1 - \frac{1^{r(3)}}{\mathbf{g}(n)^k})^{n^\varepsilon}$  such that:

- (a)  $\text{Prob}(\mathcal{M} \models \varphi_{1, i(1), i(2)}(\bar{a}_{1, m(1)}, \bar{a}_{2, m(2)}, \bar{c})) = r/\mathbf{g}(n)^k$  for some  $r \in \mathbb{R}_{>0}, h \in \mathbb{N} \setminus \{0\}$  not depending on  $n$
- (b) the probability that  $m(1), m(2) < n_4^* \Rightarrow \mathcal{M} \models \varphi_{1, i(1), i(2)}(\bar{a}_{1, m(1)}, \bar{a}_{2, m(2)}, \bar{c}) \sim (1 - q)^{n^{2\varepsilon}} \wedge (1 \sim r/\mathbf{g}(n)^k)^{n^{2\varepsilon}} \sim e^{-rn^{2\varepsilon}/g(n)^k}$ .

So this could not have occurred.

Clause (C):

Also straightforward. □1.15

\* \* \*

**Definition 1.16.** Let  $\mathbf{s}, \bar{\varphi}$  be as above.

- 1) We say  $(\mathbf{s}, \bar{\varphi})$  is reduced when: for every  $\bar{p} \in \mathbf{P}_{\mathbf{s}}^2$  and random enough  $\mathcal{M} = \mathcal{M}_{\mathbf{s}, \bar{p}, n}$  and  $\bar{c} \in \ell g(\bar{z}_{\bar{\varphi}}) \mathcal{M}$  satisfying  $\varphi_2(\bar{z}_{\bar{\varphi}})$ , the graph  $H = H_{\bar{\varphi}, \mathcal{M}, \bar{c}}$  is not  $H = H_{\bar{\varphi}', \mathcal{M}, \bar{c}'}$  when  $(\bar{\varphi}', \bar{c}')$  appropriate and  $\text{Rang}(\bar{c}') \subsetneq \text{Rang}(\bar{c})$ ; recall  $\bar{c}$  is without repetitions.
- 2) We say  $(\mathbf{s}, \bar{\varphi})$  is complete when each  $\varphi_{1, i}(\bar{x}_i, \bar{z})$  is a complete quantifier free  $\mathbb{L}(\tau_{\mathbf{s}})$ -formula realized in some  $\mathbf{s}$ -struture.

**Definition 1.17.** 1) Let  $\mathbf{s}, \bar{\varphi}', \bar{\varphi}''$  and  $\psi(\bar{z}_{\bar{\varphi}}, \bar{z}_{\bar{\varphi}'})$  be as in 1.19 and  $\bar{\varphi}', \bar{\varphi}''$  are reduced and complete. We say  $(\mathbf{s}, \bar{\varphi}^1), (\mathbf{s}, \bar{\varphi}^2)$  are explicitly isomorphic when for some  $\pi$  and  $\varkappa$  we have:

- (a)  $\mathbf{i}(\bar{\varphi}^1) = \mathbf{i}(\bar{\varphi}^2), \ell g(\bar{z}_{\bar{\varphi}^1}) = \ell g(\bar{z}_{\bar{\varphi}^2})$
- (b)  $\pi$  is a permutation of  $\{0, \dots, \mathbf{i}(\bar{\varphi}^1) - 1\}$  such that  $k_{\bar{\varphi}^1, i} = k_{\bar{\varphi}^2, \pi(i)}$  nad  $K_{\bar{\varphi}^1, i} = K_{\bar{\varphi}^2, i}$  for  $i < \bar{\varphi}^1$
- (c)  $\varkappa$  is a permutation of  $\ell g(\bar{z}_{\bar{\varphi}})$  or just one to one from  $\ell g(\bar{c}')$  onto  $< \ell g(\bar{c}^2)$
- (d) for random enough  $\mathcal{M} = \mathcal{M}_{\mathbf{s}, \bar{p}, n}$ , if  $\ell \in \{1, 2\}, M \models \varphi_2^\ell[\bar{c}_\ell]$  then letting  $\bar{c}_{3-\ell}$  be such that  $\bar{c}_2 = \varkappa(\bar{c}_1)$  we have  $M \models \varphi_2^{3-\ell}[\bar{c}_{3-\ell}]$  and  $\varphi_{1, i}(\mathcal{M}, \bar{c}_1) = \varphi_{1, \pi(i)}(\mathcal{M}, \bar{c}_2)$  and  $\varphi_{1, i, j}(\mathcal{M}, \bar{c}_1) = \varphi_{1, \pi(i), \pi(j)}(\mathcal{M}, \bar{c}_2)$

2) For  $\mathbf{s}, \bar{\varphi}$  as above let  $K_{\bar{\varphi}} = K_{\mathbf{s}, \bar{\varphi}}$  be the group of permutations  $\kappa$  of  $\ell g(\bar{z}_{\bar{\varphi}})$  such that  $\bar{\varphi}$  is explicitly isomorphic to itself using our  $\varkappa$  in 1.17(1).

**Claim 1.18.** 1) For every  $\mathbf{s}$ -scheme  $\bar{\varphi}$  we can find  $\langle \bar{\varphi}^\iota(\bar{z}_\iota) : \iota < \iota(*) \rangle$  such that:

- (a)  $\bar{\varphi}^\iota(\bar{z}_\iota)$  is a complete reduced  $\mathbf{s}$ -scheme such that  $\bar{z}^\iota$  is a subsequence of  $\bar{z}$
- (b) for every  $m$  and  $\bar{c} \in \bar{\varphi}$  for some  $\iota$  letting  $\bar{c}_\iota = \langle c_j : j \in \text{dom}(\bar{z}) \text{ and } z_1 \text{ appears in } \bar{z}^\iota \rangle$  we have  $H_{\bar{\varphi}, M, \bar{c}} = H_{\bar{\varphi}^\iota, M, \bar{c}^\iota}$
- (c) for every  $\bar{p} \in \mathbf{P}_{\mathbf{s}}^2$  and random enough  $\mathcal{M} = \mathcal{M}_{n, \bar{p}}$  and  $\iota < \iota(*)$ ,  $\bar{c}^\iota \in \varphi_2^\iota(M)$  there is  $\bar{c}$  such that  $(\bar{c}, \bar{c}^\iota, \bar{\varphi}, \bar{\varphi}^\iota)$  are as in clause (b).

2) For complete  $\bar{\varphi}$  in the definition of “trivial”, “degenerated”, “reduced” we can replace “some  $\bar{c}'$ ” by “ $\bar{c}'$ ”.

3) In the definition of  $\mathbb{L}(\mathbf{Q}_t)(\tau)$ , we can use  $(\mathbf{Q}, \dots, \bar{x}_{1, i}, \bar{x}'_{1, i}, \dots), \bar{\varphi}, \bar{z}$  onto for complete reduced non-trivial, non-degenerated.

*Proof.* Easy. □1.18

**The Isomorphism Claim 1.19.** Assume  $\mathbf{s}$  is an  $I$ -kind and complete reduced  $\bar{\varphi}', \bar{\varphi}''$  are  $\mathbf{s}$ -schemes as above.

1) For every one to one function  $\varkappa$  from  $\ell g(\bar{z}_{\bar{\varphi}'})$  onto  $\text{tp}(\bar{z}, \bar{y})$  such that  $\psi(\bar{z}_{\bar{\varphi}}, \bar{z}_{\bar{\varphi}'}) \vdash \varphi'_2(\bar{z}_{\bar{\varphi}'}) \wedge \varphi''_2(\bar{z}_{\bar{\varphi}''})$ , there is a truth value  $\mathbf{t}$  such that: if  $\mathcal{M} = \mathcal{M}_{\mathbf{s}, \bar{p}, n}$  random enough and  $\mathcal{M} \models \varphi'_2[\bar{c}'] \wedge \varphi''_2[\bar{c}'']$  so  $H' = H_{\bar{\varphi}', \mathcal{M}, \bar{c}'}, H'' = H_{\bar{\varphi}'', \mathcal{M}, \bar{c}''}$  are well defined and moreover  $M \models \psi[\bar{c}', \bar{c}'']$  then  $H' \cong H''$  iff  $\mathbf{t} = \text{truth}$  and  $\text{Rang}(\bar{c}') = \text{Rang}(\bar{c}'')$ .

2) Moreover,  $H', H''$  are isomorphic iff  $(\mathbf{s}, \bar{\varphi}'), (\mathbf{x}, \bar{\varphi}'')$  are explicitly isomorphic, see Definition 1.17 below.

3) Being explicitly isomorphic  $\mathbf{s}$ -schemes is an equivalence relation.

*Proof.* Straightforward. □1.19

## § 2. THE QUANTIFIER

**Definition 2.1.** Let  $\iota \in \{1, 2\}$ .

1) We say  $\mathbf{Q} = \mathbf{Q}_K$  is a  $\mathbf{h} - \iota$ -high-graph quantifier when :

- (a)  $\mathbf{Q}$  is a quantifier on finite graphs, i.e. it is a class of finite graphs closed under isomorphisms
- (b) if  $H$  is a finite graph and is  $\mathbf{h} - \iota$ -low then  $H \notin \mathbf{Q}$ .

2) We define a probability space on the set of high-graph quantifiers as follows: let  $\bar{H}^* = \langle H_m^* : m \in \mathbb{N} \rangle$  be a sequence of pairwise non-isomorphic finite graph such that each finite graph is isomorphic to (exactly) one of them.

For  $\iota \in \{1, 2\}$ , we let (but we may have forgot to write)

- (a)  $\mathbf{T}_\iota = \{\bar{\mathbf{t}} : \bar{\mathbf{t}} = \langle \mathbf{t}_m : m \in \mathbb{N} \rangle, \mathbf{t}_m \text{ a truth value, } \mathbf{t}_m = 0 \text{ if } H_m^* \text{ is } \mathbf{h} - \iota\text{-low}\}$
- (b) we draw the  $\mathbf{t}_m$ 's independently,  $\mathbf{t}_m = 0$  if  $H_m^*$  is  $\mathbf{i} - \iota$ -low and  $\mathbf{t}_m = 1$  has probability  $1/\mathbf{j}(n)$  when  $H_m$  is not  $\mathbf{h} - \iota$ -low
- (c) Let  $\mu_{\mathbf{t}_\iota}$  be the derived distribution.

2A) So the probability space is  $(\mathbb{B}_{\mathbf{t}_\iota}, \mu_{\mathbf{t}_\iota})$ ,  $\mathbb{B}$  is the family of Borel subsets of  ${}^{\mathbb{N}}2$ ,  $\mu_*$  the measure.

3) For  $\bar{\mathbf{t}} \in \mathbf{T}$  let  $\mathbf{Q}_{\bar{\mathbf{t}}}^\iota$  be the quantifier  $\mathbf{Q}_{K_{\bar{\mathbf{t}}}}$ ,  $K_{\bar{\mathbf{t}}} = \{H : H \text{ a finite graph isomorphic to some } H_m^* \text{ such that } \mathbf{t}_m = 1\}$ .

4) We say  $H$  is  $\mathbf{h} - \iota$ -high where  $H$  is a finite graph which is not  $\mathbf{h} - \iota$ -low.

**Claim 2.2.** For every  $\bar{\mathbf{t}} \in \mathbf{T}$ .

- 1)  $\mathbf{Q}_{\bar{\mathbf{t}}}$  is a Lindström quantifier.
- 2) For random graph  $\mathcal{G}_{n,p}$ ,  $\mathbf{Q}_{\bar{\mathbf{t}}}^\iota$  define non-trivial quantifiers, defining (without parameters) not first order definable sets.
- 3) More specifically the formula  $\psi(x) = (\text{the graph restricted to } \{y : yRx\} \text{ belongs to } K_{\bar{\mathbf{t}}})$  for every  $k$ , define in every random enough  $\mathcal{G}_{n,p}$ , a set which is not first order definable by a formula of length  $k$ .

*Proof.* Straightforward. □<sub>2.2</sub>

**Definition 2.3.** 1) So the set of formulas  $\varphi(\bar{x})$  of  $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})(\tau_{\mathbf{s}})$  for a kind sequence  $\mathbf{s}$  is the closure of the set of atomic formulas of  $\mathbb{L}(\tau_{\mathbf{s}})$  by negation ( $\psi(\bar{x}) = \neg\varphi(\bar{x})$ ), conjunction ( $\psi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$ ), existential quantification ( $\psi(\bar{x}) = (\exists y)\varphi(\bar{x}, y)$ ) and applying  $\mathbf{Q}_{\bar{\mathbf{t}}}$ ,  $\psi(\bar{z}) = (\mathbf{Q}_{\bar{\mathbf{t}}}, \dots, \bar{x}_{0,i}, \bar{x}'_{0,i}, \dots)_{i < \mathbf{i}(\bar{\varphi})} \bar{\varphi}$  where  $\bar{\varphi}$  is an  $\mathbf{s}$ -scheme of formulas which are already in  $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}}(\tau_{\mathbf{s}}))$ .

2) Satisfaction, i.e. for an  $\mathbf{s}$ -structure  $M$ , formula  $\varphi(\bar{x})$  and sequence  $\bar{a}$  of elements of  $M$  of length  $\ell g(\bar{x})$ , we define the truth vlue of  $M \models \varphi[\bar{a}]$  by induction on  $\varphi$ , the new case is when:

$$\bullet \varphi(\bar{z}_{\bar{\varphi}}) = (\mathbf{Q}_{\bar{\mathbf{t}}}, \dots, \bar{x}_{0,i}, \bar{x}'_{0,i}, \dots)_{i < \mathbf{i}(\bar{\varphi})} \bar{\varphi}.$$

Now  $M \models \varphi[\bar{c}]$  iff  $\bar{c} \in \varphi_2(M)$  and  $H_{\bar{\varphi}, M, \bar{c}}$  is isomorphic to some graph from  $\{H_m^* : \mathbf{t}_m = 1\}$ .

3) The syntax of  $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}})$  does not depend on  $\bar{\mathbf{t}}$  so may write  $\mathbb{L}(\mathbf{Q})$  that is  $\mathbb{L}(\mathbf{Q})(\tau)$  is the relevant set of formulas, but the satisfaction depends so we shall write  $M \models_{\bar{\mathbf{t}}} \varphi[\bar{a}]$  for  $\bar{a}$  a sequence from  $M$  and formula  $\varphi(\bar{x}) \in \mathbb{L}(\mathbf{Q})$ . Of course  $\ell g(\bar{a}) = \ell g(\bar{x})$ .

**Theorem 2.4.** 1) For any  $j \in (0, 1)_{\mathbb{R}}$  for all but a null set of  $\bar{\mathbf{t}} \in \mathbf{T}$ , the random graph  $\mathcal{G}_{n,p}$  satisfies the 0-1 law for the logic  $\mathbb{L}(\mathbf{Q}_{\bar{\mathbf{t}}}^j)$ , i.e. we may allow to apply  $\mathbf{Q}_{\bar{\mathbf{t}}}$  to definitions as in Definition 1.6, see Claim 2.4.

2) Moreover, the limit theory does not depend on  $\bar{\mathbf{t}}$ .

*Remark 2.5.* Of course, we can replace the class of graphs by the class of  $\mathbf{s}$ -structures,  $\mathbf{s}$  any kind sequence.

*Proof.* Consider a sentence  $\psi \in \mathbb{L}(\mathbf{Q})$ , see 2.3

$\boxplus_0$  for each  $n$  we consider drawing  $(\mathcal{G}_{n,p}, \bar{\mathbf{t}}) \in \text{Graph}_n \times \mathbf{T}$ , that is, independently we draw

- $\bar{\mathbf{t}} \in \mathbf{T}$  by the probability space from 2.1(2)
- $\mathcal{G}_{n,p} \in \text{Graph}_n$  = the set of graphs with set of nodes  $[n]$  with each edge drawn with probability  $p_n$  independently of the other edges

$\boxplus_1$  It suffices to prove that

- (a) the probability of “ $\mathcal{G}_{n,p} \models_{\bar{\mathbf{t}}} \psi$ ”, i.e. the pair  $(\mathcal{G}_{n,p}, \bar{\mathbf{t}})$  satisfies this, either is  $\geq \frac{1}{2^{nr}}$  or is  $\geq 1 - \frac{1}{2^{nr}}$  for some  $r = r(\psi) \in (0, 1)_{\mathbb{R}}$
- (b) which case does not depend on  $n$ .

[Why? For every  $\psi \in \mathbb{L}(\mathbf{Q})$ , the following events  $\mathcal{E}_{\psi}^1 \wedge \mathcal{E}_{\psi}^2$  has probability zero, where

$$\mathcal{E}_{\psi}^1 := (\text{for infinitely many } n, \mathcal{G}_{n,p} \models_{\bar{\mathbf{t}}} \psi)$$

$$\mathcal{E}_{\psi}^2 := (\text{for infinitely many } n, \mathcal{G}_{n,p} \models_{\bar{\mathbf{t}}} \neg \psi).$$

This holds by (a)+(b) of  $\boxplus_1$ . Hence also the event  $\mathcal{E} = \bigvee \{\mathcal{E}_{\psi}^1 \wedge \mathcal{E}_{\psi}^2 : \psi \in \mathbb{L}(\mathbf{Q})\}$  has probability zero. Hence, by Fubini theorem, drawing for a set of  $\bar{\mathbf{t}}$ 's of measure 1, the event  $\mathcal{E}_{\varphi}^1[\bar{\mathbf{t}}] \wedge \mathcal{E}_{\psi}^2[\bar{\mathbf{t}}]$  has probability zero, where  $\mathcal{E}_{\psi}^{\ell}[\bar{\mathbf{t}}]$  is the event  $\mathcal{E}_{\psi}^{\ell}$  fixing  $\bar{\mathbf{t}}$ .

We can find a  $\bar{\Delta}$  such that

- $\boxplus_2$  (a)  $\bar{\Delta} = \langle \Delta_{\ell} : \ell \leq \ell(*) \rangle$
- (b)  $\Delta_{\ell}$  is a finite set of formulas from  $\mathbb{L}(\mathbf{Q})$  increasing with  $\ell$
- (c)  $\Delta_0$  is the set of quantifier free formulas
- (d)  $\psi \in \Delta_{\ell(*)}$
- (e) every formula in  $\Delta_{2\ell+1} \setminus \Delta_{2\ell}$  is gotten from formulas from  $\Delta_{2\ell}$  by first order operation  $(\neg \varphi(\bar{x}), \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x}), \exists y \varphi(\bar{x}, y))$
- (f) every formula in  $\Delta_{2\ell+2} \setminus \Delta_{2\ell+1}$  is of the form  $\psi(\bar{z}) = (\mathbf{Q} \dots \bar{x}_i, \bar{x}'_i, \dots)_{i < k} \bar{\varphi}(\bar{z})$  where  $\bar{\varphi} = \bar{\varphi}(\bar{z})$  recalling 1.19 is a complete reduced  $\mathbf{s}$ -scheme for some  $\mathbf{s}$ , i.e. is as in Definition 1.6 but the  $\varphi_{0,i}(\bar{x}, \bar{z}), \varphi_{1,i,j}(\bar{x}_i, \bar{x}_j, \bar{z}), \varphi_2(\bar{z})$  being from  $\Delta_{2\ell+1}$
- (g) no two  $\bar{\varphi}$ 's which occur in  $\Delta$  on  $(\mathbf{Q}, \dots) \bar{\varphi}$  are explicitly isomorphic (see 1.17).

[Why? Should be clear.]

- $\boxplus_3$  let  $\Delta_{\ell} = \{\vartheta_s(\bar{x}_s) : s \in I_{\ell}^*\}$  recalling  $m < \ell \Rightarrow I_m^* \subseteq I_{\ell}^*$  and  $I_0 = I_s$

Now by induction on  $\ell \leq \ell(*)$  we choose  $\mathbf{s}_\ell, \bar{\vartheta}'_\ell, \bar{\vartheta}''_\ell$  and the function  $G \mapsto M_{G, \bar{\mathbf{t}}, \ell}$  for  $G$  a graph on  $[n]$  some  $n$  such that:

- $\boxplus_{4, \ell}(A)$  (a)  $I_\ell$  finite  
 (b)  $\mathbf{s}_\ell$  is as in Definition 1.3, an  $I_\ell$ -kind of a vocabulary  
 (c)( $\alpha$ )  $\mathbf{s}, I_0$  are defined by  $I_0 = \{s_0\}$  for some  $s_0 \notin I_{\ell(*)}^*$   
 $n_{\mathbf{s}, s_0} = 2, K_{\mathbf{s}_0, s_0} = \text{Sym}(2)$ , the group of permutation of  $\{0, 1\}$   
 ( $\beta$ )  $I_{2\ell+1} = I_{2\ell}$   
 ( $\gamma$ )  $I_{2\ell+2} = I_{2\ell+1} \cup (I_{2\ell+2}^* \setminus I_{2\ell+1}^*)$   
 ( $\delta$ ) so  
 •  $\langle I_\ell : \ell \leq \ell(*) \rangle$  is increasing  
 •  $\mathbf{M}_{\mathbf{s}_0, n}$  is  $\text{Graph}_n$ , the set of graphs with set of nodes  $[n]$   
 (d)  $\bar{\vartheta}'_\ell = \langle \vartheta'_s(\bar{x}_s) : s \in I_\ell \rangle$   
 (e)  $\vartheta'_s(\bar{x}_s)$  a formula in  $\mathbb{L}(\tau_{\mathbf{s}_\ell})$  for  $s \in I_\ell$   
 (f)  $\vartheta''_s(\bar{x}_s)$  is a quantifier free formula in  $\mathbb{L}(\tau_{\mathbf{s}_\ell})$  equivalent to  $\vartheta'_s(\bar{x}_s)$  in the limit theory  $T_{\mathbf{s}_\ell}$ , see 1.4  
 (g) for any given  $G \in \mathcal{G}_{n, p}$ , i.e.  $G \in \mathbf{M}_{\mathbf{s}_0, n}$  and  $\bar{\mathbf{t}} \in \mathbf{T}$   
 we define  $M_{G, \bar{\mathbf{t}}, \ell} \in \mathbf{M}_{\mathbf{s}_\ell, n}$  by:  
 ( $\alpha$ )  $M_{G, \bar{\mathbf{t}}, \ell}$  is a  $\tau_{\mathbf{s}_\ell}$ -model expanding  $M_{G, \bar{\mathbf{t}}, m}$  for  $m < \ell$  and for  
 $s \in I_\ell, R_s^{M_{G, \bar{\mathbf{t}}, \ell}}_{\mathbf{s}_\ell}$  is defined by  $\vartheta_s(x_s)$  and also by  $\vartheta'_s(x_s)$   
 ( $\beta$ ) if  $\ell = 0, M_{G, \bar{\mathbf{t}}, \ell}$  is  $G$   
 ( $\gamma$ ) if  $\ell = 2m + 1, s \in I_\ell \setminus I_{2m}$  we apply the the first order  
 construction of  $\vartheta_s(\bar{x}_s)$  from the formulas  $\langle \vartheta_t(\bar{x}_t) : t \in I_{2m} \rangle$   
 to construct  $\vartheta'_s(\bar{x}_s)$  from  $\langle \vartheta''_t(\bar{x}_t) : t \in I_{2m} \rangle$   
 ( $\delta$ ) for  $\ell = 2m + 2$  and  $s \in I_\ell \setminus I_{2m+1}$  if  $\vartheta_s(\bar{x}_s) = (\mathbf{Q} \dots, \bar{x}_i, \bar{x}'_i, \dots)_{i < i(\bar{\varphi}_i)} \bar{\varphi}_s(\bar{x}_s)$   
 we define  $\vartheta'_s(\bar{x}_s)$  by replacing in  $\bar{\varphi}_s$  every  $\vartheta_t$  by  $\vartheta''_t$  getting  $\bar{\varphi}'_s$   
 and let  $\vartheta'_s(\bar{x}_s) = (\mathbf{Q} \dots, \bar{x}_i, \bar{x}'_i, \dots)_{i < i(\bar{\varphi}_i)} \bar{\varphi}'_s$   
 ( $\varepsilon$ ) we choose  $\vartheta''_i(\bar{x}_i)$  by 1.4 sequence clause (f) here.

Now for each  $\ell \leq \ell(*)$  we have two relevant ways to draw as  $\mathbf{s}_\ell$ -structure  $M$  with-universe = set of elements  $[n]$ .

First, draw  $\mathbf{t} \in \mathbf{T}$  and  $\mathcal{G} = \mathcal{G}_{q, n}$  (recall  $q \in (0, 1)_{\mathbb{R}}$  was fixed in the beginning of Theorem 2.4) and compute  $M_{\mathcal{G}, \bar{\mathbf{t}}, n}$ , a  $\mathbf{s}_\ell$ -structure. This induces a distribution  $\mu_{q, n, \ell}$  on  $\mathbf{M}_{\mathbf{s}_\ell, n}$ , i.e.  $\mu_{q, n, \ell}(M) = \text{Prob}(M_{\mathcal{G}, \bar{\mathbf{t}}, n} = M | \mu_{\text{gr}, q, n} \times \mu_{\mathbf{T}} = \mu_{\text{gr}, \bar{p}, q, n} \times \mu_{\mathbf{T}})$ .

Second, we shall choose  $\bar{p}_\ell$  in  $\mathbf{P}_{\mathbf{s}_\ell}^2$  and draw  $\mathcal{M}_{\mathbf{s}, \bar{p}_\ell, n}$  here the distribution is ? The interest in the first is that our aim is to prove the 0-1 law for  $M_{\mathcal{G}, \bar{p}, n}$ , in particular, for  $\ell = \ell(*)$  and our sentence  $\psi$ ; we use the other  $\ell$ 's in an induction.

A priori the probability of  $M_{\mathcal{G}, \bar{p}, n} \models \psi$  is opaque.

For the second,  $\mathcal{M}_{\mathbf{s}, \bar{p}_\ell, n}$  an understanding of the probability of  $M_{\mathbf{s}, \bar{p}_\ell, n} \models \psi$  is now well known and satisfies the 0-1 law. Hence it suffices to prove that the distribution of  $M_{\mathcal{G}, \bar{\mathbf{t}}, \ell}$  (for  $\mathcal{G} \in \mathcal{G}_{p, n}$ ) from  $\mathbf{M}_{\mathbf{s}_\ell, n}$  and  $M_{\mathbf{s}_\ell, \bar{p}, n} \in \mathbf{G}_{\mathbf{s}_\ell, n}$  are sufficiently similar.

Naturally we choose:

- (\*)<sub>1</sub> (a)  $p_{\mathbf{s}_{\text{gr}}, \mathbf{s}_{\text{gr}}, n} = p_{\mathbf{s}_0, \mathbf{s}_{\text{gr}}, n} = q$   
 (b)  $p_{\mathbf{s}_\ell, t, n} = q / \mathbf{g}(n)$  for  $t \in I_{\mathbf{s}_\ell} \setminus \{s_{\text{sq}}\}$ .

Of course, we induct one for  $\ell = 0$  there is no difference so we deal now with  $\ell + 1$  if  $\ell$  is even this is trivial so assume  $\ell$  is odd.

There are several reasons for a difference, for a given model  $M \in \mathbf{M}_{\mathbf{s}_\ell, n}$

- (\*)<sub>M,2</sub>  $t \in I_{\ell+1}^* \setminus I_\ell^*$  and  $\bar{c} \in \varphi_{t,2}(M)$ . The graph  $H_{\bar{\varphi}_t, M, \bar{c}}$  is  $\iota$ -low (for a given  $n$  there are at most  $n^{k(\bar{\varphi}_t)}$  (check cases)
- (\*)<sub>M,2</sub> for some  $t(1), t(2) \in I_{\ell+1}^* \setminus I_\ell^*$ ,  $\bar{c}_2 \in \varphi_{t(1),2}(M)$  and  $\bar{c}_2 \in \varphi_{t,2}(M)$  we have  $(t(j), \bar{c}_1/E_{\bar{\varphi}'_{t(1)}})_\varphi \neq (t(2), \bar{c}_2/E_{\bar{\varphi}'_{t(2)}})$  but the graphs  $H_{\bar{\varphi}'_{t(1)}, M, \bar{c}_1}, H_{\bar{\varphi}'_{t(2)}, M, \bar{c}_2}$  are isomorphic
- (\*)<sub>M,3</sub> for some  $t(1), t(2) \in I_{\ell+2}^* \setminus I_\ell^*$  and  $t(2) \in \cup\{I_{2k+2}^* \setminus I_{2k+1}^* : 2k+2 \leq \ell\}$  and  $\bar{c}_1 \in \varphi_{t(1),2}(M), \bar{c}_2 \in \varphi'_{t(2),2}(M)$  the graphs  $H_{\bar{\varphi}'_{t(1)}, M, \bar{c}_2}, H_{\bar{\varphi}'_{t(2)}, M, \bar{c}_2}$  are isomorphic
- (\*)<sub>M,4</sub> the sequence  $\bar{p} \in \mathbf{P}_q^2$  try to immitate  $\mathbf{t}$ , but having the probability for  $\mathcal{M}_{\mathbf{s}_{\ell+1}, \bar{p}, n} \models R_t[\bar{c}]$  is  $p_{t,n} = 1/g(n)$  whereas the probability  $\mathbf{t}_i = 1$  is  $1/g(|H_i^*|)$  where  $i$  is such that  $H_{\bar{\varphi}_t, \mathcal{M}_{\mathcal{G}, \mathbf{t}, \bar{c}}} = H_i^*$  for  $\mathcal{G} = G_{q,n}$ .

Now there is no reason that usually  $i = n$ . However, if  $\iota = 2$  then  $|H_i^*| \leq k(\bar{\varphi}_t) \cdot n$  and if  $\iota = 1$ ,  $H_1^* \leq n^{k(\bar{\varphi}_2)}$ . In both cases with probability very close to 1, (for  $\mu_{\mathbf{s}_{\ell+1}, \bar{p}, n}$ ),  $|H_i^*| \geq n/2^{k(\bar{\varphi}_t)}$ . So clearly as  $\mathbf{q}$  grow slowly enough, see 1.1(2).

This is also true for  $(*)_{M,1}, (*)_{M,2}, (*)_{M,3}$ . Together, we have two distributions on  $\mathbf{M}_{\mathbf{s}_{\ell+1}, n}$  and for the second, omitting a set of  $M$  with small probability (in  $\mu_{\mathbf{s}_{\ell+1}, \bar{p}, n}$ ) for any other  $M$ , the two distributions give almost the same values. The computations are easy so we are done.  $\square_{2.4}$

*Remark 2.6.* To eliminate  $(*)_4$  in the end of the proof we may complicate the drawing of  $\mathcal{M}_{\mathbf{s}_{\ell+1}, \bar{p}, n}$  by? We draw  $\mathcal{M}_{\mathbf{s}_m, \bar{p}, n}$  by induction on  $m$ : if  $m = 2j + 2$ ,  $M = M_{\mathbf{x}_{2j+1}, \bar{p}, n}$  given for  $R_t(t \in I_m^* \setminus I_{2k+1}^*)$  we consider only  $\bar{c} \in \text{varphi}'_{t,2}(M)$  let  $m = m_t(\bar{c}) = m_t(\bar{c}, M)$  be the number of nodes of  $H_{\bar{\varphi}'_t, M, t}$  and we draw a truth value of  $R_t(\bar{c})$  with probability  $1/g(m)$ . Proving the 0-1 law for such drawing is easy.



§ 3. HOW TO GET A REAL QUANTIFIER, I.E. DEFINABLE  $K$ 

**Discussion 3.1.** One which seems easiest while not unreasonable is: given a finite graph  $G$ , with  $m$  points, which is reasonable - defined as in [Sh:F1166] and a point  $b$  in it, compute the valency minus  $m/2$ , divided by square root of  $m$  (or the variance of the related normal distribution) and ask if rounding to integers is odd or even.

We may replace the valency by the number of edges of  $G$ .

What are the dangers? As we may define a variant of the graph omitting one edge, in some cases this will change the truth value. For each node the probability goes to zero but in binomial distribution the probability of e.g. getting valency exactly half of the expected value (rounded) is about 1 divided by the square root of  $m$ .

So we should divide not by the square root of  $m$  but by a larger value (maybe instead of asking on even/odd of the rounded value just ask if it can be larger than one, or absolute value) such that:

- (a) almost surely (i.e. with large probability) for some node the value is above 1
- (b) the probability that it is exactly one for some node is negligible, and this is true even if we use a graph only definable (reversing edge/non-edge, omitting some, etc.).

So we should say that clearly by continuity considerations there are such choices. A danger is that the  $n$  being odd/even can be expressed.

Another avenue is to choose the more natural “the valency is at least half”; but then it seems we can express being even/odd: say change by one edge change the truth value and this is true even if we omit one node. So the number of neighborhoods is half in both cases.

## REFERENCES

- [Be85] Jon Barwise and Solomon Feferman (editors), *Model-theoretic logics*, Perspectives in Mathematical Logic, Springer Verlag, Heidelberg-New York, 1985.
- [ShSp:304] Saharon Shelah and Joel Spencer, *Zero-one laws for sparse random graphs*, Journal of the American Mathematical Society **1** (1988), 97–115.
- [Sh:F1166] Saharon Shelah, *Random Graphs: stronger logic but with the 0-1 law*.

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL, AND, DEPARTMENT OF MATHEMATICS, HILL CENTER - BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA

*E-mail address:* `shelah@math.huji.ac.il`

*URL:* `http://shelah.logic.at`